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Abstract. In this paper we prove a sharp inequality of Szegő-Weinberger type for the first nontrivial eigenvalue of the $p-$Laplacian with Neumann boundary conditions. This applies to convex sets with given diameter. Some variants, extensions and limit cases are investigated as well.

Contents

1. Introduction

1.1. **Overview.** Given an open bounded connected Lipschitz set $\Omega \subset \mathbb{R}^N$ and an exponent $1 < p < \infty$, we consider λ_p and μ_p the first nontrivial eigenvalues of the $p-$Laplace operator with Dirichlet and Neumann boundary conditions, respectively. We remind that these can

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be variationally characterized by

$$\lambda_p(\Omega) = \min_{v\in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p\, dx \,:\, \int_\Omega |v|^p\, dx = 1 \right\},$$

and

$$\mu_p(\Omega) = \min_{v\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p\, dx \,:\, \int_\Omega |v|^p\, dx = 1 \text{ and } \int_\Omega |v|^{p-2}\, v\, dx = 0 \right\}.$$

Since exact values of such quantities are known only for specific values of p and special domains Ω, it is important to give (sharp) estimates for these quantities in terms of (simple) geometric quantities such as measure, perimeter, diameter, relative isoperimetric constants and so on. In this direction, the reader could consult for instance [1, 2, 3, 4, 10] and the references therein.

With this respect the most celebrated example is the *Faber–Krahn inequality* (see [11, Chapter 3] for example) which asserts that the following minimization problem

$$\inf\{\lambda_p(\Omega) \,:\, |\Omega| \le c\}, \tag{1.1}$$

is (uniquely) solved by $N-$dimensional balls of measure c. By taking advantage of the homegeneity properties of the functional $\Omega \mapsto \lambda_p(\Omega)$, the previous can be summarized as

$$\lambda_p(\Omega) \ge \lambda_p(B) \left(\frac{|B|}{|\Omega|}\right)^{\frac{p}{N}}, \tag{1.2}$$

where B is now any $N-$dimensional ball. Then $\lambda_p(\Omega)$ can be bounded from below in a sharp way just in terms the measure of the set Ω. We point out that by using *isoperimetric inequality* or *isodiametric inequality*, from (1.2) we can infer similar lower bounds for λ_p in terms of the perimeter or the diameter of Ω.

Observe that problem (1.1) becomes trivial, when we replace $\lambda_p(\Omega)$ with $\mu_p(\Omega)$. Indeed, the latter is actually zero each time Ω is disconnected. It turns out that the natural counterpart for μ_p is rather the maximization problem, i.e.

$$\sup\{\mu_p(\Omega) \,:\, |\Omega| \ge c\}. \tag{1.3}$$

Again, this is generally expected to be solved by $N-$dimensional balls of volume c. Unfortunately so far this problem has resisted all the attempts to be attacked with the unique exception of the case $p = 2$ (and partially of the limiting cases $p = 1$ and $p = \infty$, see [7] and [8, 18] respectively). The *Szegő–Weinberger inequality* [19, 23] states in fact that for $p = 2$ problem (1.4) is (uniquely) solved by $N-$dimensional balls of measure c. As before, the result can be rewritten in scaling invariant form as

$$\mu_2(\Omega) \le \mu_2(B) \left(\frac{|B|}{|\Omega|}\right)^{\frac{2}{N}}, \tag{1.4}$$

with equality holding if and only if Ω is an $N-$dimensional ball. We recall that the proof of (1.4) for $p = 2$ crucially exploits some pecularities of the Laplacian, like linearity and the knowledge of the explicit form of eigenfunctions on balls.

A couple of comments on the Szegő-Weinberger result are in order. First of all, inequality (1.4) says that $\mu_2(\Omega)$ can be estimated *from above* just in terms of the measure of Ω. But differently from the case λ_p, now *we can not directly infer* similar upper bounds for $\mu_2(\Omega)$ in terms of perimeter or diameter. Then one may wonder whether such a kind of estimates hold true or not for every $1 < p < \infty$, at least for some particular classes of sets.

Secondly, we notice that if B is any $N-$dimensional ball, using the fact that $\mu_2(B) < \lambda_2(B)$ (see [11] or Proposition 5.1 below), from (1.4) we can infer

$$\mu_2(\Omega) \leq \lambda_2(B) \left(\frac{|B|}{|\Omega|}\right)^{\frac{2}{N}}, \tag{1.5}$$

which can be see as a weak version of the Szegő-Weinberger inequality. Again, a natural question is whether inequality (1.5) can be extended to the case of $p \neq 2$ or not.

The last two questions are the starting point of our analysis. In this paper we prove indeed sharp upper bounds on $\mu_p(\Omega)$ in terms of diameter, as well as generalizations of (1.5) for $p \neq 2$, under the additional constraint that Ω is a convex set.

1.2. **A sharp upper bound.** Then our main scope is to investigate the following shape optimization problem with convexity and diameter constraints

$$\sup\{\mu_p(\Omega) \,:\, \Omega \subset \mathbb{R}^N \text{ convex, } \mathrm{diam}(\Omega) \geq 1\}. \tag{1.6}$$

Of course, by homogeneity of the quantities involved the value 1 has no bearing and could be replaced by any constant $c > 0$. In Theorem 3.1 we show that the previous upper bound μ^* is finite, then we compute it and show at the same time that *this problem does not admit a solution.* Notably, we show that for every admissible set Ω there holds

$$\mu_p(\Omega) < \mu^*,$$

and we exhibit a sequence $\{\Omega_n\} \subset \mathbb{R}^N$ of convex sets suitably degenerating to a segment for which

$$\lim_{n\to\infty} \mu_p(\Omega_n) = \mu^*.$$

We refer to Section 3 for more details. As we will show, the previous result can be summarized by the following scaling invariant sharp inequality

$$\mu_p(\Omega) < \lambda_p(B) \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^p, \tag{1.7}$$

where B is any $N-$dimensional ball and inequality sign is strict. The proof of (1.7) is based on a clever choice of a special test function which reminds idea exploited in [19, 23]. We point out that by joining (1.7) and the *isodiametric inequality*, we immediately get

$$\mu_p(\Omega) < \lambda_p(B) \left(\frac{|B|}{|\Omega|}\right)^{\frac{p}{N}},$$

which generalizes (1.5) to $p \neq 2$ for convex sets, as announced above. By keeping in mind the way such an estimate was proved for $p = 2$, the previous can be seen as *the trace of a potentially exhisting Szegő-Weinberger inequality for the $p-$Laplacian.*

For ease of completeness and in order to neatly motivate some of the studies performed in this paper, it is useful to recall at this point that the *minimization* problem

$$\inf\{\mu_p(\Omega)\,:\,\Omega\subset\mathbb{R}^N \text{ convex, } \mathrm{diam}(\Omega)\le 1\}, \tag{1.8}$$

highlights the same features as problem (1.6). For example, here as well the infimum can be computed and is not attained. More interestingly, a minimizing sequence is again given by a family of convex sets collapsing on a segment. For $p=2$ this is a celebrated result by Payne and Weinberger (see [16]), which has been recently generalized in [9, 22] to $p\neq 2$. The result can be summarized by the sharp inequality

$$\mu_p(\Omega) > \left(\frac{\pi_p}{\mathrm{diam}(\Omega)}\right)^p. \tag{1.9}$$

1.3. **Generalized eigenvalues.** It is quite natural to wonder if similar conclusions can be drawn also in the case of the following generalized notion of eigenvalues

$$\lambda_{p,q}(\Omega)=\min_{v\in W^{1,p}_0(\Omega)}\left\{\int_\Omega |\nabla v|^p\,dx\,:\,\int_\Omega |v|^q\,dx=1\right\},$$

and

$$\mu_{p,q}(\Omega)=\min_{v\in W^{1,p}(\Omega)}\left\{\int_\Omega |\nabla v|^p\,dx\,:\,\int_\Omega |v|^q\,dx=1\ \text{ and }\ \int_\Omega |v|^{q-2}\,v\,dx=0\right\}.$$

Quite interestingly, it turns out that for $q>p$ one has the following picture:

- one can prove the analogue of (1.7);
- this estimate is not sharp;
- the maximization problem (1.6) now admits a solution;
- a lower bound like (1.9) is not possible (and the infimum in (1.8) is 0);

On the contrary, for $q<p$ all the previous statements have to be reverted. In particular, we have

$$\sup\{\mu_{p,q}(\Omega)\,:\,\Omega\subset\mathbb{R}^N \text{ convex, } \mathrm{diam}(\Omega)\ge 1\}=+\infty,$$

and it is rather the minimization problem for $\mu_{p,q}$ which is now well-posed (see Section 4 for more details).

1.4. **Plan of the paper.** In Section 2 we prove some basic results concerning properties of $\mu_{p,q}(\Omega)$ and $\lambda_{p,q}(\Omega)$. Section 3 is devoted to the investigation of problem (1.6). In Section 4 we consider the case $1<p<q<p^*$, and we prove a lower bound for $\mu_{p,q}(\Omega)$ in terms of measure and diameter of Ω when $p>q$. In Section 5 we prove a nodal domain property which roughly speaking shows that for $q>p$ eigenfunctions associated to $\mu_{p,q}(\Omega)$ can not have a closed nodal line. Finally, the last Section is devoted to investigate the limit cases $q=p^*$ and $p=N$.

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2. Preliminaries

We fix two exponents p and q such that $1 < q, p < \infty$ and

$$q < p^* = \begin{cases} \dfrac{N\,p}{N-p}, & \text{if } 1 < p < N, \\ \\ +\infty, & \text{if } p \geq N. \end{cases} \tag{2.1}$$

For every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, we use the standard Sobolev spaces

$$W^{1,p}(\Omega) = \big\{u \in L^p(\Omega) \,:\, \nabla u \in L^p(\Omega;\mathbb{R}^N)\big\},$$

and $W^{1,p}_0(\Omega)$, the latter being the closure of $C^\infty_0(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$. We then define the two quantities

$$\mu_{p,q}(\Omega) = \inf_{v\in W^{1,p}(\Omega)\setminus\{0\}} \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\left(\displaystyle\int_\Omega |v|^q\,dx\right)^{\frac{p}{q}}} \,:\, \int_\Omega |v|^{q-2}\,v\,dx = 0 \right\},$$

and

$$\lambda_{p,q}(\Omega) = \inf_{v\in W^{1,p}_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\left(\displaystyle\int_\Omega |v|^p\,dx\right)^{\frac{p}{q}}}.$$

It is useful to recall that $\mu_{p,q}(\Omega)$ can be defined through the unconstrained minimization

$$\mu_{p,q}(\Omega) = \inf_{v\in \widehat{W}^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\min_{t\in\mathbb{R}}\left(\int_\Omega |v-t|^q\,dx\right)^{\frac{p}{q}}},$$

where we set $\widehat{W}^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) \,:\, \int_\Omega |\nabla v|^p\,dx > 0\}$. Also, we have that if λ is such that the equation

$$-\Delta_p u = \lambda\,\|u\|^{p-q}_{L^q(\Omega)}\,|u|^{q-2}\,u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

admits a nontrivial solution in $W^{1,p}_0(\Omega)$, then $\lambda \geq \lambda_{p,q}(\Omega)$.

Remark 2.1. Observe that the quantity $\mu_{p,q}(\Omega)$ is always well-defined, we could have $\mu_{p,q}(\Omega) = 0$ if Ω does not support a Poincaré inequality of the type

$$C\,\min_{t\in\mathbb{R}}\left(\int_\Omega |v-t|^q\,dx\right)^{\frac{p}{q}} \leq \int_\Omega |\nabla v|^p\,dx.$$

The same remark applies to the quantity $\lambda_{p,q}(\Omega)$.

We start with a couple of preliminary results on the quantities $\mu_{p,q}$ and $\lambda_{p,q}$.

Lemma 2.2. *Let $1<p<\infty$ and $1<s<q<p^*$, then we have*

$$\lambda_{p,q}(\Omega)\le|\Omega|^{\frac{p}{s}-\frac{p}{q}}\,\lambda_{p,s}(\Omega)\qquad \textit{and}\qquad \mu_{p,q}(\Omega)\le|\Omega|^{\frac{p}{s}-\frac{p}{q}}\,\mu_{p,s}(\Omega).$$

Proof. The result is a plain consequence of Hölder inequality. Let us prove for example the second inequality: we pick $u_s\in\widehat{W}^{1,p}(\Omega)$ a function minimizing the Rayleigh quotient which defines $\mu_{p,s}(\Omega)$. We then define t_q as the minimizer of

$$t\mapsto\int_\Omega|u_s-t|^q\,dx,$$

thus we get

$$\begin{aligned}\mu_{p,q}(\Omega)=\min_{v\in\widehat{W}^{1,p}(\Omega)}\frac{\displaystyle\int_\Omega|\nabla v|^p\,dx}{\displaystyle\min_{t\in\mathbb{R}}\left(\int_\Omega|v-t|^q\,dx\right)^{\frac{p}{q}}}&\le\frac{\displaystyle\int_\Omega|\nabla u_s|^p\,dx}{\displaystyle\left(\int_\Omega|u_s-t_q|^q\,dx\right)^{\frac{p}{q}}}\\&\le|\Omega|^{\frac{p}{s}-\frac{p}{q}}\frac{\displaystyle\int_\Omega|\nabla u_s|^p\,dx}{\displaystyle\left(\int_\Omega|u_s-t_q|^s\,dx\right)^{\frac{p}{s}}}\le|\Omega|^{\frac{p}{s}-\frac{p}{q}}\frac{\displaystyle\int_\Omega|\nabla u_s|^p\,dx}{\displaystyle\min_{t\in\mathbb{R}}\left(\int_\Omega|u_s-t|^s\,dx\right)^{\frac{p}{s}}}\end{aligned}$$

which in turn gives

$$\mu_{p,q}(\Omega)\le|\Omega|^{\frac{p}{s}-\frac{p}{q}}\,\mu_{p,s}(\Omega),$$

as desired. □

The following simple continuity result will be useful.

Lemma 2.3. *Let $1<p<\infty$ and $1<q<p^*$, then we have*

$$\lim_{s\to q}\lambda_{p,s}(\Omega)=\lambda_{p,q}(\Omega)\qquad \textit{and}\qquad \lim_{s\to q}\mu_{p,s}(\Omega)=\mu_{p,q}(\Omega).$$

Proof. We just prove the second equality, the first one can be proved along the same lines. Let $u_q\in W^{1,p}(\Omega)\setminus\{0\}$ be a minimizer for the variational problem defining $\mu_{p,q}(\Omega)$, i.e. such that

$$\frac{\displaystyle\int_\Omega|\nabla u_q|^p\,dx}{\displaystyle\left(\int_\Omega|u_q|^q\,dx\right)^{\frac{p}{q}}}=\mu_{p,q}(\Omega),\qquad\text{and}\qquad\int_\Omega|u_q|^{q-2}\,u_q\,dx=0.$$

Observe that since $u_q\neq0$, the second condition above implies that u_q is not constant. We then define t_s to be the minimizer of

$$t\mapsto\int_\Omega|u_q-t|^s\,dx,\qquad t\in\mathbb{R},$$

thus we obtain

$$\begin{aligned}\limsup_{s\to q}\mu_{p,s}(\Omega) &\le \limsup_{s\to q}\frac{\displaystyle\int_\Omega |\nabla u_q|^p\,dx}{\displaystyle\min_{t\in\mathbb{R}}\left(\int_\Omega |u_q-t|^s\,dx\right)^{\frac{p}{s}}}\\ &= \limsup_{s\to q}\frac{\displaystyle\int_\Omega |\nabla u_q|^p\,dx}{\displaystyle\left(\int_\Omega |u_q-t_s|^s\,dx\right)^{\frac{p}{s}}} = \frac{\displaystyle\int_\Omega |\nabla u_q|^p\,dx}{\displaystyle\left(\int_\Omega |u_q|^q\,dx\right)^{\frac{p}{q}}} = \mu_{p,q}(\Omega).\end{aligned}$$

where we used that[1] t_s goes to 0 as s goes to q. In order to prove that

$$\liminf_{s\to q}\mu_{p,s}(\Omega)\ge\mu_{p,q}(\Omega),$$

as well, we have to distinguish two cases. If $s\nearrow q$, then we can simply apply Lemma 2.2, thus we obtain

$$\liminf_{s\nearrow q}\mu_{p,s}(\Omega)=\liminf_{s\nearrow q}|\Omega|^{\frac{p}{s}-\frac{p}{q}}\,\mu_{p,s}(\Omega)\ge\mu_{p,q}(\Omega).$$

On the contrary, for $s\searrow q$, we pick $u_s\in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_s|^p\,dx=\mu_{p,s}(\Omega),\qquad \int_\Omega |u_s|^s\,dx=1,\qquad \int_\Omega |u_s|^{s-2}\,u_s=0,$$

then in particular $\{u_s\}$ is a bounded sequence in $W^{1,p}(\Omega)$, thus there exists a subsequence (not relabeled) which strongly converges when $s\searrow q$ in $L^q(\Omega)$. If we call u this limit function, we have

$$\int_\Omega |u|^p=\lim_{s\searrow q}\int_\Omega |u_s|^s\,dx=1\qquad\text{and}\qquad\int_\Omega |u|^{q-2}\,u\,dx=\lim_{s\searrow q}\int_\Omega |u_s|^{s-2}\,u_s\,dx=0$$

and of course

$$\mu_{p,q}(\Omega)\le\int_\Omega|\nabla u|^p\,dx\le\liminf_{s\searrow q}\int_\Omega|\nabla u_s|^p\,dx=\liminf_{s\searrow q}\mu_{p,s}(\Omega).$$

This concludes the proof. □

We will also need the following very simple geometric result for convex sets.

[1] The convex functions

$$t\mapsto\int_\Omega |u-t|^s\,dx,$$

are converging locally uniformly to the function

$$t\mapsto\int_\Omega |u-t|^q\,dx.$$

Moreover, for $|s-q|$ sufficiently small, there exists $m>0$ such that we have $|t_s|<m$. Then by uniform convergence and uniqueness of the minimizer t_q we can infer $\lim_{s\to q}t_s=t_q=0$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open convex set, and let $x_0 \in \partial\Omega$. Then*

$$\langle x - x_0, \nu_\Omega(x)\rangle \geq 0, \qquad \text{for } \mathcal{H}^{N-1}-\text{a.e. } x \in \partial\Omega,$$

where $\nu_\Omega(x)$ denotes the outer unit normal to $\partial\Omega$ at the point x.

Proof. Since Ω is convex, given $x \in \partial\Omega$ we have that

$$\overline{\Omega} \subset \{y \in \mathbb{R}^N \,:\, \langle y - x, \nu_\Omega(x)\rangle \leq 0\},$$

i.e. the hyperplane orthogonal to $\nu_\Omega(x)$ and passing from x is a supporting hyperplane for Ω. In particular, since $x_0 \in \overline{\Omega}$ we get

$$\langle x_0 - x, \nu_\Omega(x)\rangle \leq 0,$$

which concludes the proof. □

3. A Szegő-Weinberger inequality for convex sets

The following is the main result of the paper. This shows that the nonlinear spectral optimization problem

$$\sup\{\mu_{p,p}(\Omega) \,:\, \Omega \subset \mathbb{R}^N \text{ convex, } \mathrm{diam}(\Omega) \geq 1\},$$

does not admit a solution, but a maximizing sequence is given by a family of convex sets suitably degenerating to a segment. Of course, the value 1 for the diameter constraint plays no special role and could be replaced by any constant $c > 0$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set and $1 < p < \infty$. Then we have*

$$\mu_{p,p}(\Omega) < \lambda_{p,p}(B) \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^p, \tag{3.1}$$

where B is any $N-$dimensional ball.

Equality sign in (3.1) *is never achieved but the inequality is sharp. More precisely, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ of convex sets such that:*

- $\mathrm{diam}(\Omega_k) = d > 0$, *for every $k \in \mathbb{N}$;*
- *Ω_k converges to a segment of length d in the Hausdorff topology;*
- *we have*

$$\lim_{k\to\infty} \mu_{p,p}(\Omega_k) = \lambda_{p,p}(B_{d/2}), \tag{3.2}$$

where $B_{d/2}$ is an $N-$dimensional ball having radius $d/2$.

Proof. We split the proof into two parts: at first we prove (3.1), then we construct the sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ verifying (3.2).

Proof of (3.1). First of all, we observe that inequality (3.1) is in scaling invariant form. Then without loss of generality, we can confine ourselves to prove that

$$\mu_{p,p}(\Omega) < \lambda_{p,p}(B),$$

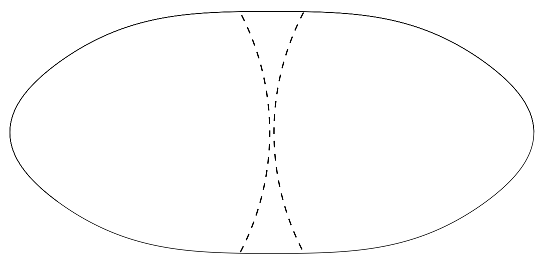

FIGURE 1. The construction of the two caps Ω_0 and Ω_1.

where B is the ball centered at the origin such that $\mathrm{diam}(\Omega) = \mathrm{diam}(B)$. Let us take $u \in C^{1,\alpha}(\overline{B}) \cap C^\infty(B \setminus \{0\})$ the first Dirichlet eigenfunction for the ball B, normalized by the condition $u > 0$. This solves

$$-\Delta_p u = \lambda_{p,p}(B)\, u^{p-1} \qquad \text{and} \qquad u = 0 \quad \text{on } \partial B. \tag{3.3}$$

We then take two points $x_0, x_1 \in \partial\Omega$ such that $|x_0 - x_1| = \mathrm{diam}(\Omega)$, and we define the two caps

$$\Omega_i = \left\{ x \,:\, |x - x_i| < \frac{\mathrm{diam}(\Omega)}{2} \right\} \cap \Omega, \qquad i = 0, 1,$$

which are mutually disjoint (see Figure 1). We then take the function

$$\varphi(x) = u(x - x_0) \cdot 1_{\Omega_0}(x) - c\, u(x - x_1) \cdot 1_{\Omega_1}(x) \in W^{1,p}(\Omega),$$

where $c \in \mathbb{R}$ is the constant given by

$$c = \frac{\displaystyle\int_{\Omega_0} u(x - x_0)^{p-1}\, dx}{\displaystyle\int_{\Omega_1} u(x - x_1)^{p-1}\, dx}, \qquad \text{so that} \qquad \int_\Omega |\varphi|^{p-2}\, \varphi\, dx = 0.$$

By using this function φ in the Rayleigh quotient defining $\mu_{p,p}(\Omega)$, we get

$$\mu_{p,p}(\Omega) < \frac{\displaystyle\int_{\Omega_0} |\nabla u(x - x_0)|^p\, dx + c^p \int_{\Omega_1} |\nabla u(x - x_1)|^p\, dx}{\displaystyle\int_{\Omega_0} |u(x - x_0)|^p\, dx + c^p \int_{\Omega_1} |u(x - x_1)|^q\, dx},$$

where the strict inequality holds since φ can not be an eigenfunction[2]. By performing an integration by parts in the integrals at the numerator, we obtain[3]

$$\begin{aligned}\int_{\Omega_0} |\nabla u(x-x_0)|^p\,dx &= \int_{\partial\Omega_0} |\nabla u(x-x_0)|^{p-2}\,\frac{\partial u}{\partial \nu_\Omega}(x-x_0)\,u(x-x_0)\,d\mathcal{H}^{N-1}(x)\\ &\quad-\int_{\Omega_0} \Delta_p u(x-x_0)\,u(x-x_0)\,dx\\ &= \int_{\partial\Omega_0} |\nabla u(x-x_0)|^{p-2}\,\frac{\partial u}{\partial \nu_\Omega}(x-x_0)\,u(x-x_0)\,d\mathcal{H}^{N-1}(x)\\ &\quad+\lambda_{p,p}(B)\int_{\Omega_0} |u(x-x_0)|^p\,dx,\end{aligned}$$

where we used the equation (3.3) solved by u. Observe that the first integral in the right-hand side has a sign. Indeed u is a radially decreasing function, then (with a small abuse of notation) we have

$$\langle \nabla u(x-x_0), \nu_\Omega)(x)\rangle = u'(|x-x_0|)\,\left\langle \frac{x-x_0}{|x-x_0|}, \nu(x)\right\rangle,$$

and the claim follows from Lemma 2.4. The same computations apply to the other terms appearing in the numerator, thus obtaining

$$\mu_{p,p}(\Omega) < \lambda_{p,p}(B)\,\frac{\displaystyle\int_{\Omega_0} |u(x-x_0)|^q\,dx + c^p\int_{\Omega_1} |u(x-x_1)|^p\,dx}{\displaystyle\int_{\Omega_0} |u(x-x_0)|^p\,dx + c^p\int_{\Omega_1} |u(x-x_1)|^q\,dx} = \lambda_{p,p}(B),$$

which concludes the proof of (3.1).

Optimality of (3.1). Let B be a ball of diameter d. We prove optimality of (3.1) by constructing a sequence of convex sets $\{\Omega_k\}_{k\in\mathbb{N}}$, all sharing the same diameter d, and such that

$$\lambda_{p,p}(B) \le \liminf_{k\to\infty} \mu_{p,p}(\Omega_k) \qquad \text{and} \qquad \limsup_{k\to\infty} \mu_{p,p}(\Omega_k) \le \lambda_{p,p}(B). \tag{3.4}$$

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[2]Observe that if the Rayleigh quotient of φ achieves $\mu_{p,q}(\Omega)$, then φ would solve

$$-\Delta_p\varphi = \mu_{p,q}(\Omega)\,|\varphi|^{q-2}\,\varphi, \qquad \text{in } \Omega,$$

in a weak sense. Let us take $y_0 \in \partial\Omega_0\cap\Omega$, by picking a ball $B_\varrho(y_0)$ with radius ϱ sufficiently small so that $B_\varrho(y_0)\subset\Omega\setminus\Omega_1$, we would obtain that φ is a nonnegative solution of the equation in $B_\varrho(y_0)$. Then by Harnack's inequality (see [21, Theorem 1.1]) one obtains

$$0 < \max_{B_\varrho(y_0)} \varphi \le C\,\min_{B_\varrho(y_0)} \varphi = 0,$$

thus getting a contradiction. We point out that *we are not using any unique continuation argument.*

[3]Observe that we have $u(x-x_0)=0$ on $\partial\Omega_0\cap\Omega$.

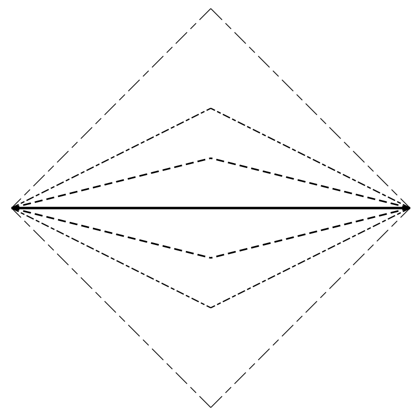

FIGURE 2. The maximizing sequence Ω_k of Theorem 3.1.

In view of (3.1) we only need to prove the liminf inequality in (3.4). For all $s \in \mathbb{R}$ and $k \in \mathbb{N}$ let us denote by

$$\mathcal{C}_k^-(s) = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, (x_1 - s)_- > 2\,k\,|x'|\right\}$$

and

$$\mathcal{C}_k^+(s) = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, (x_1 - s)_+ > 2\,k\,|x'|\right\}$$

the left and right circular infinite cone in $\mathbb{R}^N$ whose axis is the x_1-axis, having vertex in $(s, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, and whose opening angle is $\alpha = 2\,\arctan\left(\frac{1}{k}\right)$. We set

$$\Omega_k = \mathcal{C}_k^-\left(\frac{d}{2}\right) \cap \mathcal{C}_k^+\left(-\frac{d}{2}\right). \tag{3.5}$$

In dimension $N = 2$, Ω_k is nothing but a rhombus of diagonals d and $1/k$. In higher dimension Ω_k is obtained by gluing together the basis of two right circular cones of height $d/2$ and radii $1/(2\,k)$ (see Figure 2).

We claim that for this family (3.4) holds true. We start observing that whenever u belongs to $W^{1,p}(\Omega_k)$ then the rescaled function $v(x_1, x') = u\,(x_1, x'/k)$ belongs to $W^{1,p}(\Omega_1)$ and we have

$$\int_{\Omega_1} \left(|\partial_{x_1} v|^2 + k^2 |\nabla_{x'} v|^2\right)^{\frac{p}{2}}\, dx = k^{N-1} \int_{\Omega_k} |\nabla u|^p\, dx, \qquad \int_{\Omega_1} |v|^p\, dx = k^{N-1} \int_{\Omega_k} |u|^p\, dx,$$

and

$$\int_{\Omega_1} |v|^{p-2}\, v\, dx = k^{N-1} \int_{\Omega_k} |v|^{p-2}\, v\, dx = 0.$$

Thus we obtain

$$
\begin{aligned}
\mu_{p,p}(\Omega_k) &= \min_{u\in W^{1,p}(\Omega_k)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{\Omega_k} |\nabla u|^p\,dx}{\displaystyle\int_{\Omega_k} |u|^p\,dx} \,:\, \int_{\Omega_k} |u|^{p-2}\,u\,dx = 0 \right\}, \\
&= \min_{v\in W^{1,p}(\Omega)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{\Omega_1} \left(|\partial_{x_1} v|^2 + k^2\,|\nabla_{x'} v|^2\right)^{\frac{p}{2}}\,dx}{\displaystyle\int_{\Omega_1} |v|^p\,dx} \,:\, \int_{\Omega_1} |v|^{p-2}\,v\,dx = 0 \right\} =: \gamma_k(\Omega_1).
\end{aligned}
$$

Now we denote by u_k a function which minimizes the Rayleigh quotient defining $\mu_{p,p}(\Omega_k)$ and by $v_k(x_1,x') = u_k\,(x_1, x'/k)$ the corresponding function which minimizes the functional defining $\gamma_k(\Omega)$. Without loss of generality we can assume that $\|v_k\|_{L^p(\Omega)} = 1$. Since inequality (3.1) implies that

$$
\int_{\Omega} \left(|\partial_{x_1} v_k|^2 + k^2\,|\nabla_{x'} v_k|^2\right)^{\frac{p}{2}}\,dx \le C_{N,p,d}, \qquad \text{for all } k\in\mathbb{N}, \tag{3.6}
$$

then there exists $w\in W^{1,p}(\Omega)\setminus\{0\}$ so that $v_k \to w$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Moreover we also have[4]

$$
\nabla_{x'} w \equiv 0, \qquad \text{and} \qquad \int_{\Omega} |w|^{p-2}\,w\,dx = 0.
$$

Since w does not depend on the x' variable, we will write for simplicity $w = w(x_1)$ with a slight abuse of notation. For all $s\in[-d/2, d/2]$ we denote by Γ_s the section of Ω which is

[4]The bound (3.6) implies that for every given $k_0\in\mathbb{N}$, we have

$$
k_0^p \int_{\Omega} |\nabla_{x'} w|^p\,dx \le \int_{\Omega} \left(|\nabla_{x_1} w|^2 + k_0^2\,|\nabla_{x'} w|^2\right)^{\frac{p}{2}}\,dx \le \liminf_{k\to\infty} \int_{\Omega} \left(|\nabla_{x_1} v_k|^2 + k_0^2\,|\nabla_{x'} v_k|^2\right)^{\frac{p}{2}}\,dx \le C,
$$

which in turn gives $\nabla_{x'} w \equiv 0$ by the arbitrariness of k_0.

orthogonal to the x_1-axis at $x_1 = s$ and set $g(s) = \mathcal{H}^{N-1}(\Gamma_s)$. Then we get

$$\begin{aligned}
\liminf_{k\to\infty} \gamma_k(\Omega) &= \liminf_{k\to\infty} \frac{\displaystyle\int_{\Omega_1} \left(|\partial_{x_1} v_k|^2 + k^2 |\nabla_{x'} v_k|^2\right)^{\frac{p}{2}}\, dx}{\displaystyle\int_{\Omega_1} |v_k|^p\, dx} \ge \liminf_{k\to\infty} \frac{\displaystyle\int_{\Omega_1} |\partial_{x_1} v_k|^p\, dx}{\displaystyle\int_{\Omega_1} |v_k|^p\, dx} \\
&\ge \frac{\displaystyle\int_{\Omega_1} |w'|^p\, dx}{\displaystyle\int_{\Omega_1} |w|^p\, dx} = \frac{\displaystyle\int_{-d/2}^{d/2} |w'(s)|^p\, g(s)\, ds}{\displaystyle\int_{-d/2}^{d/2} |w(s)|^p\, g(s)\, ds} \\
&\ge \min_{f\in W^{1,p}\left(-\frac{d}{2},\frac{d}{2}\right)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{-d/2}^{d/2} |f'|^p\, g\ ds}{\displaystyle\int_{-d/2}^{d/2} |f|^p\, g\, ds} \ :\ \int_{-d/2}^{d/2} |f|^{p-2}\, f\, g\ ds = 0 \right\}.
\end{aligned}$$

Let us denote by η the previous minimal value, then a minimizer f is a solution to the following boundary value problem

$$\begin{cases} -\left(g\,|f'|^{p-2}\,f'\right)' = \eta\, g\,|f|^{p-2}\,f, & \text{in } (-d/2, d/2), \\[2mm] f'(-d/2) = f'(d/2) = 0 \end{cases}$$

Since $g(s) = g(-s)$ it is easy to prove that $f(0) = 0$ and hence f solves

$$\begin{cases} -\left(g\,|f'|^{p-2}\,f'\right)' = \eta\, g\,|f|^{p-2}\,f, & \text{in } (0, d/2), \\[2mm] f(0) = f'(d/2) = 0 \end{cases}$$

Finally, by reminding that $g(s) = \omega_{N-1}(s - d/2)^{N-1}$, if we set $h(r) = f(d/2 - r)$ then

$$\begin{cases} -\left(r^{N-1}\,|h'|^{p-2}\,h'\right)' = \eta\, r^{N-1}\,|h|^{p-2}\,h, & \text{in } (0, d/2), \\[2mm] h'(0) = h(d/2) = 0 \end{cases}$$

which means that $H(x) = h(|x|)$ is a Dirichlet eigenfunction of $-\Delta_p$ of a N-dimensional ball of radius $d/2$, namely B. Hence $\eta \ge \lambda_{p,p}(B)$ and we get

$$\liminf_{k\to\infty} \gamma_k(\Omega) \ge \lambda_{p,p}(B),$$

which concludes the proof. $\square$

4. The case $p \neq q$

In this section we discuss variants and extensions of Theorem 3.1 for the quantity $\mu_{p,q}$ when $p \neq q$.

4.1. **The case $p < q$.** Actually, with the very same proof of Theorem 3.1 we can prove the following upper bound (see Remark 4.2 below for a discussion on its sharpness).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set and $1 < p < q < p^*$. Then we have*

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(B) \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^{p+\frac{N\,p}{q}-N}, \tag{4.1}$$

where B is any $N-$dimensional ball.

Proof. We use the same notation as in the proof of Theorem 3.1. With $u \in C^{1,\alpha}(\overline{B}) \cap C^\infty(B \setminus \{0\})$ we now indicate the function achieving $\lambda_{p,q}(B)$, normalized by the conditions

$$\|u\|_{L^q(B)} = 1 \qquad \text{and} \qquad u > 0. \tag{4.2}$$

Though we do not need this, we recall that such a function is unique for B and radially symmetric decreasing (see [6, Main Theorem]). Moreover it solves

$$-\Delta_p u = \lambda_{p,q}(B)\, u^{q-1} \qquad \text{and} \qquad u = 0 \quad \text{on } \partial B. \tag{4.3}$$

As before, we consider the two caps Ω_0 and Ω_1 and take

$$\varphi(x) = u(x - x_0) \cdot 1_{\Omega_0}(x) - c\, u(x - x_1) \cdot 1_{\Omega_1}(x) \in W^{1,p}(\Omega),$$

where $c \in \mathbb{R}$ is now given by

$$c = \frac{\displaystyle\int_{\Omega_0} u(x - x_0)^{q-1}\, dx}{\displaystyle\int_{\Omega_1} u(x - x_1)^{q-1}\, dx}.$$

By using this function φ in the Rayleigh quotient defining $\mu_{p,q}(\Omega)$ and proceeding as in Theorem 3.1 we now end up with

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(B)\, \frac{\displaystyle\int_{\Omega_0} |u(x - x_0)|^q\, dx + c^p \int_{\Omega_1} |u(x - x_1)|^q\, dx}{\left(\displaystyle\int_{\Omega_0} |u(x - x_0)|^q\, dx + c^q \int_{\Omega_1} |u(x - x_1)|^q\, dx\right)^{\frac{p}{q}}}.$$

Observe that the term on the right-hand side is of the form

$$\frac{A + t^{\frac{p}{q}}\, B}{(A + t\, B)^{\frac{p}{q}}}.$$

For $p < q$ the previous expression is maximal for $t = 1$. By observing that this maximal value is given by $(A + B)^{1-p/q}$, we thus get

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(B) \left[\int_{\Omega_0} |u(x - x_0)|^q\, dx + \int_{\Omega_1} |u(x - x_1)|^q\, dx\right]^{1-\frac{p}{q}}. \tag{4.4}$$

We have $1 - p/q > 0$ and the sum of the two terms into square brackets is less than 1 by (4.2), thus we can finally infer (3.1). $\square$

Remark 4.2 (About sharpness)**.** This time, the estimate (4.1) is not sharp. We keep the same notation as in the proof of Theorem 4.1 and still consider $q > p$. By adding and subtracting the term $\lambda_{p,q}(B)$ on the right-hand side of (4.4), recalling (4.2) and using the concavity of $t \mapsto t^{1-p/q}$, we get

$$\begin{aligned}\mu_{p,q}(\Omega) &< \lambda_{p,q}(B) + \lambda_{p,q}(B)\left[\left(\int_{\Omega_0\cup\Omega_1} |u|^q\,dx\right)^{1-\frac{p}{q}} - \left(\int_B |u|^q\,dx\right)^{1-\frac{p}{q}}\right]\\ &\le \lambda_{p,q}(B) + \frac{q-p}{q}\,\lambda_{p,q}(B)\left[\int_{\Omega_0\cup\Omega_1} |u|^q\,dx - \int_B |u|^q\,dx\right]\\ &\le \lambda_{p,q}(B) - \frac{q-p}{q}\,\lambda_{p,q}(B)\int_{B\setminus T_\Omega} |u|^q\,dx,\end{aligned}$$

Since u is radially decreasing, a simple rearrangement argument finally gives

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(B) - \frac{q-p}{q}\,\lambda_{p,q}(B)\int_{B\setminus T_\Omega} |u|^q\,dx \tag{4.5}$$

where T_Ω is the ball centered at the origin, such that $|T_\Omega| = |\Omega_0 \cup \Omega_1|$. Now observe that by using the *quantitative isodiametric inequality* (see [14, Theorem 1])

$$\begin{aligned}|B \setminus T_\Omega| = |B| - |\Omega_0 \cup \Omega_1| &\ge |\Omega| + \frac{|\Omega|}{C_N}\,\mathcal{A}(\Omega)^2 - |\Omega_0 \cup \Omega_1|\\ &= \Big||\Omega| - |\Omega_0 \cup \Omega_1|\Big| + \frac{|\Omega|}{C_N}\,\mathcal{A}(\Omega)^2,\end{aligned}$$

where $C_N > 0$ is a dimensional constant and $\mathcal{A}(\Omega)$ is the *Fraenkel asymmetry* of Ω, defined by

$$\mathcal{A}(\Omega) = \inf\left\{\frac{2\,|\Omega \setminus \Omega_\#|}{|\Omega_\#|}\ :\ \Omega_\# \text{ ball with } |\Omega_\#| = |\Omega|\right\}.$$

The estimate (4.5) shows that there can not exist a convex set Ω such that $\mu_{p,q}(\Omega) = \lambda_{p,q}(B)$, for $q > p$. Indeed, it this were true, one would obtain

$$\int_{B\setminus T_\Omega} |u|^q\,dx = 0,$$

and since $u > 0$ in B, this would imply $|B \setminus T_\Omega| = 0$ and thus

$$\mathcal{A}(\Omega) = 0 \qquad \text{and} \qquad |\Omega| = |\Omega_0 \cup \Omega_1|. \tag{4.6}$$

The first condition in (4.6) implies that Ω is a ball, in contrast with the fact that $|\Omega| > |\Omega_0 \cup \Omega_1|$ for a ball (see Figure 3).

From Theorem 3.1, we can also infer a couple of upper bounds on $\mu_{p,q}$, in terms of measure and diameter.

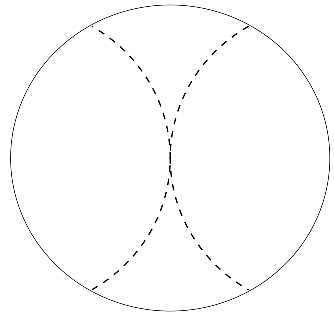

FIGURE 3. The two caps Ω_0 and Ω_1 can not cover the whole ball.

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set and $1 < p \le q < p^*$. Then we have*

$$\mu_{p,q}(\Omega) < \lambda_{p,p}(B)\, |\Omega|^{1-\frac{p}{q}} \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^p, \tag{4.7}$$

and also

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(B) \left(\frac{|B|}{|\Omega|}\right)^{\frac{p}{N}+\frac{p}{q}-1}, \tag{4.8}$$

where B is any $N-$dimensional ball.

Proof. We first observe that (4.7) with $p = q$ coincides with (3.1). Let $p < q$, by applying Lemma 2.2 we get

$$\mu_{p,q}(\Omega) \le |\Omega|^{1-\frac{p}{q}}\, \mu_{p,p}(\Omega).$$

It is then sufficient to apply Theorem 3.1 with $q = p$ in order to estimate the right-hand side. This proves (4.7).

In order to prove (4.8), we can use the isodiametric inequality

$$\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)} \le \left(\frac{|B|}{|\Omega|}\right)^{\frac{1}{N}}, \tag{4.9}$$

in equations (3.1) and (4.1). □

Remark 4.4. From estimate (4.7), we have that for $p < q$ the quantity $\mu_{p,q}(\Omega)$ can not be bounded *from below* in terms on $\mathrm{diam}(\Omega)$ only. In other words, for $q > p$ we have

$$\inf\{\mu_{p,q}(\Omega) \,:\, \Omega \subset \mathbb{R}^N \text{ convex, } \mathrm{diam}(\Omega) \le c\} = 0.$$

Indeed, for any sequence of convex sets $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\lim_{k\to\infty} |\Omega_k| = 0 \qquad \text{and} \qquad \mathrm{diam}(\Omega_k) = c,$$

we have that $\mu_{p,q}(\Omega_k)$ converges to 0, as k tends to ∞.

As in the case $p = q$, we can then ask whether the following shape optimization problem

$$\sup\{\mu_{p,q}(\Omega)\,:\,\Omega \text{ open and bounded convex set, } \mathrm{diam}(\Omega) \geq c\}, \tag{4.10}$$

admits a solution or not. We have the following existence result.

Theorem 4.5 (Existence of a maximizer)**.** *Let $1 < p < q < p^*$, for every $c > 0$ problem* (4.10) *admits a solution. In other words, there exists an open and bounded convex set $\mathcal{K} \subset \mathbb{R}^N$ such that*

$$\mu_{p,q}(\Omega)\,\Big(\mathrm{diam}(\Omega)\Big)^{p+\frac{N\,p}{q}-N} \leq \mu_{p,q}(\mathcal{K})\,\Big(\mathrm{diam}(\mathcal{K})\Big)^{p+\frac{N\,p}{q}-N},$$

for every $\Omega \subset \mathbb{R}^N$ open and bounded convex set.

Proof. By Theorem 3.1 we already know that this supremum is finite. Let us call N_c this supremum and take a maximizing sequence of admissible sets $\{\Omega\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$, of course we can suppose that

$$\mu_{p,q}(\Omega_k) \geq \frac{N_c}{2} > 0, \qquad \text{for every } k \in \mathbb{N}. \tag{4.11}$$

Since $\mu_{p,q}$ scales like a length to a negative power, we can also assume that

$$\mathrm{diam}(\Omega_k) = c, \qquad \text{for every } k \in \mathbb{N}.$$

Finally, thanks to (4.7) we can suppose that there exists a uniform constant $\delta > 0$ such that

$$|\Omega_k| \geq \delta, \qquad \text{for every } k \in \mathbb{N}, \tag{4.12}$$

since otherwise we would have that $\mu_{p,q}(\Omega_k)$ goes to zero.

Thanks to the bound on the diameters, we can assume that the whole sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ is contained in a common compact set $D \subset \mathbb{R}^N$. Thus the sequence is relatively compact for the complementary Hausdorff topology in D: more precisely, there exists an open set $\Omega \subset D$ such that Ω_k (up to a subsequence) converges in the Hausdorff complementary distance to Ω (see [12, Corollaire 2.2.24]). Moreover, Ω is still convex and its diameter equals c (see [12, Section 2.2.3]). We also observe that the characteristic functions $\{1_{\Omega_k}\}_{k\in\mathbb{N}}$ converges to 1_Ω strongly[5] in $L^1(D)$ and $*-$weakly in $L^\infty(D)$.

Without loss of generality, we can assume that Ω contains the origin, since $\mu_{p,q}(\Omega)$ is not affected by translations. We are now going to prove that

$$\limsup_{k\to\infty} \mu_{p,q}(\Omega_k) \leq \mu_{p,q}(\Omega). \tag{4.13}$$

At this aim, let us take $u \in W^{1,p}(\Omega)$ a function attaining the infimum in the definition of $\mu_{p,q}(\Omega) > 0$. Since Ω contains the origin, for every $\varepsilon > 0$ the set $\Omega_\varepsilon = (1+\varepsilon)\,\Omega$ is such that

$$\Omega \Subset \Omega_\varepsilon,$$

[5]By convexity, a uniform bound on $\mathrm{diam}(\Omega_k)$ implies a uniform bound on their perimeters and measures. Then it is sufficient to use the compact embedding $BV(D) \hookrightarrow L^1(D)$.

then by Hausdorff convergence for every $\varepsilon>0$ there exists $k_\varepsilon\in\mathbb{N}$ such that

$$\Omega_k\subset\Omega_\varepsilon,\qquad \text{for every } k\ge k_\varepsilon.$$

We also set

$$u_\varepsilon(x)=u\left(\frac{x}{1+\varepsilon}\right),\qquad x\in\Omega_\varepsilon,$$

then for every $0<\varepsilon<1$ and every $k\ge k_\varepsilon$, we take $t_{\varepsilon,k}\in\mathbb{R}$ such that

$$\int_{\Omega_k}|u_\varepsilon-t_{\varepsilon,k}|^q\,dx=\min_{t\in\mathbb{R}}\int_{\Omega_k}|u_\varepsilon-t|^q\,dx,$$

We claim that the sequence $\{t_{\varepsilon,k}\}_{k\in\mathbb{N}}$ in bounded uniformly in k and $0<\varepsilon<1$, i.e. there exists $C>0$ such that

$$|t_{\varepsilon,k}|\le C,\qquad \text{for every}\quad 0<\varepsilon<1\quad\text{and}\quad k\ge k_\varepsilon. \tag{4.14}$$

Indeed, observe that by convexity of the map $\tau\mapsto\tau^q$ and (4.12), we have

$$\begin{aligned}\int_{\Omega_k}|u_\varepsilon-t_{\varepsilon,k}|^q\,dx&\ge\frac{1}{2^{q-1}}\,|\Omega_k|\,|t_{\varepsilon,k}|^q-\int_{\Omega_k}|u_\varepsilon|^q\,dx\\&\ge\frac{\delta}{2^{q-1}}\,|t_{\varepsilon,k}|^q-\int_{\Omega_\varepsilon}|u_\varepsilon|^q\,dx\\&=\frac{\delta}{2^{q-1}}\,|t_{\varepsilon,k}|^q\,-(1+\varepsilon)^N\int_\Omega|u|^q\,dx,\end{aligned}$$

and on the other hand

$$\int_{\Omega_k}|u_\varepsilon-t_{\varepsilon,k}|^q\,dx\le\frac{\displaystyle\int_{\Omega_k}|\nabla u_\varepsilon|^p\,dx}{\mu_{p,q}(\Omega_k)}\le 2\,\frac{(1+\varepsilon)^{N-p}}{N_c}\int_\Omega|\nabla u|^p\,dx,$$

where we used (4.11) and the very definition fo u_ε. By keeping the two estimates together, we finally get (4.14).

Thus we can suppose that $t_{\varepsilon,k}$ converges (up to a subsequence) to $t_\varepsilon\in\mathbb{R}$ as k goes to ∞, and t_ε is in turn uniformly bounded. Then we get

$$\limsup_{k\to\infty}\mu_{p,q}(\Omega_k)\le\limsup_{k\to\infty}\frac{\displaystyle\int_{\Omega_k}|\nabla u_\varepsilon|^p\,dx}{\left(\displaystyle\int_{\Omega_k}|u_\varepsilon-t_{\varepsilon,k}|^q\,dx\right)^{\frac{p}{q}}}\le\frac{\displaystyle\int_{\Omega}|\nabla u_\varepsilon|^p\,dx}{\left(\displaystyle\int_{\Omega}|u_\varepsilon-t_{\varepsilon}|^q\,dx\right)^{\frac{p}{q}}}$$

for every $0<\varepsilon<1$, where we also used the $*-$weak convergence of the characteristic functions, recalled above. We now observe that

$$\lim_{\varepsilon\to0}\int_\Omega|u_\varepsilon-t_\varepsilon|^q\,dx=\int_\Omega|u-\widetilde{t}|^q\,dx\ge\min_{t\in\mathbb{R}}\int_\Omega|u-t|^q\,dx,$$

where $\widetilde{t}\in\mathbb{R}$ is an accumulation point of the sequence $\{t_\varepsilon\}_{\varepsilon>0}$, and also

$$\lim_{\varepsilon\to0}\|\nabla u_\varepsilon-\nabla u\|_{L^p(\Omega)}=0.$$

Thus it is now sufficient to take the limit as ε goes to 0 in order to get (4.13). This finally gives that Ω is a solution of (4.10). $\square$

4.2. **The case $p > q$.** In this case, we can show that an upper bound on $\mu_{p,q}$ like that of (4.1) can not hold true and actually we have

$$\sup\{\mu_{p,q}(\Omega)\,:\,\Omega\subset\mathbb{R}^N \text{ convex},\ \mathrm{diam}(\Omega)\ge c\}=+\infty.$$

Indeed, for every sequence of open convex sets $\{\Omega_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that

$$\mathrm{diam}(\Omega_n)=c>0 \qquad \text{and} \qquad \lim_{n\to\infty}|\Omega_n|=0,$$

we have

$$\lim_{n\to\infty}\mu_{p,q}(\Omega_n)=+\infty.$$

Actually, this is a consequence of the following estimate.

Proposition 4.6. *Let $1<q<p$ and $\Omega\subset\mathbb{R}^N$ be an open and bounded convex set. Then we have*

$$\left(\frac{\pi_p}{\mathrm{diam}(\Omega)}\right)^p |\Omega|^{\frac{q}{p}-1}\le\mu_{p,q}(\Omega), \tag{4.15}$$

where the constant π_p is given by

$$\pi_p=2\int_0^{(p-1)^{\frac{1}{p}}}\left(1-\frac{t^p}{p-1}\right)^{-\frac{1}{p}}\,dt.$$

Proof. Again by Lemma 2.2 with $s=p>q$, we get

$$\mu_{p,p}(\Omega)\le|\Omega|^{1-\frac{q}{p}}\,\mu_{p,q}(\Omega).$$

By using the following lower bound on $\mu_{p,p}(\Omega)$ (see [9, Theorem 1.1])

$$\mu_{p,p}(\Omega)\ge\left(\frac{\pi_p}{\mathrm{diam}(\Omega)}\right)^p,$$

we can then conclude $\square$

By combining (4.15) with the isodiametric inequality (4.9), we get the following counterpart of Theorem 4.1 for the case $q<p$

$$\mu_{p,q}(\Omega)\ge\left(\frac{\pi_p}{|B|^{\frac{1}{q}-\frac{1}{p}}\,\mathrm{diam}(B)}\right)^p\left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^{p+\frac{N\,p}{q}-N}, \tag{4.16}$$

for every $\Omega\subset\mathbb{R}^N$ open and bounded convex set (as always B denotes any $N-$dimensional ball). Thus this time it is the minimum problem

$$\inf\{\mu_{p,q}(\Omega)\,:\,\Omega\subset\mathbb{R}^N \text{ convex},\ \mathrm{diam}(\Omega)\le c\},$$

that actually makes sense. By proceeding as in the proof of Theorem 4.5, it is not difficult to see that the previous problem admits indeed a solution.

5. A nodal domain property

If u is a function achieving the infimum in the problem defining $\mu_{p,q}(\Omega)$, then by *nodal domain* we mean every connected component of the sets

$$\{x \in \Omega : u(x) > 0\} \qquad \text{and} \qquad \{x \in \Omega : u(x) < 0\}.$$

As a consequence of Theorems 3.1 and 4.1, in the case $p \geq q$ we have the following result.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set and $1 < p \leq q < p^*$. Then*

$$\mu_{p,q}(\Omega) < \lambda_{p,q}(\Omega). \tag{5.1}$$

Moreover, every nodal domain of a function achieving $\mu_{p,q}(\Omega)$ has to intersect $\partial\Omega$.

Proof. The proof of (5.1) immediately follows from (3.1) in connection with the Faber-Krahn inequality, i.e.

$$|B|^{\frac{p}{q}+\frac{p}{N}-1}\,\lambda_{p,q}(B) \leq |\Omega|^{\frac{p}{q}+\frac{p}{N}-1}\,\lambda_{p,q}(\Omega)$$

and the isodiametric inequality.

To prove the second assertion, let us argue by contradiction. We take v achieving $\mu_{p,q}(\Omega)$ and we assume that the open set $\{x \in \Omega : v > 0\}$ has a connected component ω which is compactly contained in Ω. We can further suppose that $\|v\|_{L^q(\Omega)} = 1$, then $v \in W^{1,p}_0(\omega)$ and it solves

$$-\Delta_p v = \mu_{p,q}(\Omega)\, v^{q-1}, \qquad \text{in } \omega,$$

so that

$$\int_\omega |\nabla v|^p = \mu_{p,q}(\Omega) \int_\omega |v|^q\,dx \leq \mu_{p,q}(\Omega) \left(\int_\omega |v|^q\,dx\right)^{\frac{p}{q}},$$

thanks to the fact that

$$1 = \int_\Omega |v|^q\,dx \geq \int_\omega |v|^q\,dx,$$

and $p/q \leq 1$. This implies that

$$\lambda_{p,q}(\omega) \leq \mu_{p,q}(\Omega).$$

By using the strict monotonicity of $\lambda_{p,q}(\Omega)$ with respect to set inclusion and (5.1), we then get

$$\lambda_{p,q}(\Omega) < \lambda_{p,q}(\omega) \leq \mu_{p,q}(\Omega) < \lambda_{p,q}(\Omega),$$

which gives the desired contradiction. □

Remark 5.2. When $p = q = 2$, the previous argument to infer that first nontrivial Neumann eigenfunctions can not have a closed nodal line was originally due to Pleijel (see [15]). For the Laplacian, inequality (5.1) was conjectured by Kornhauser and Stakgold (see [13]) and then proved by Szegő and Weinberger as a consequence of their celebrated inequality (1.4).

6. Limit cases

6.1. **Sub-conformal case.** We consider $1 < p < N$ and for an open bounded set $\Omega \subset \mathbb{R}^N$ we introduce the limit quantity

$$\mu_{p,p^*}(\Omega) = \inf_{u \in W^{1,p} \setminus \{0\}} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\left(\displaystyle\int_\Omega |u|^{p^*} \, dx\right)^{\frac{p}{p^*}}} \, : \, \int_\Omega |u|^{p^*-2} \, u \, dx = 0 \right\}.$$

We also set

$$T_{N,p} = \min_{u \in W_0^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{p^*} \, dx\right)^{\frac{p}{p^*}}},$$

i.e. $T_{N,p}$ is the best constant in the Sobolev inequality for $W_0^{1,p}(\mathbb{R}^N)$ (see for instance [20]). We recall that the previous minimum is (uniquely) attained by functions of the form

$$x \mapsto c\, U\left(\frac{x - x_0}{\lambda}\right), \qquad c \in \mathbb{R} \setminus \{0\}, \, x_0 \in \mathbb{R}^N, \, \lambda > 0,$$

where U is the C^∞ decreasing function

$$U(\varrho) = \frac{1}{\left(1 + \varrho^{p/(p-1)}\right)^{\frac{N-p}{p}}}, \qquad \varrho \geq 0. \tag{6.1}$$

The following result is well-known. We provide a proof for the reader's convenience.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure. Then*

$$\lim_{q \nearrow p^*} \lambda_{p,q}(\Omega) = T_{N,p}. \tag{6.2}$$

Proof. At first we notice that by Lemma 2.2

$$\lambda_{p,p^*}(\Omega) \leq |\Omega|^{\frac{p}{q} - \frac{p}{p^*}} \, \lambda_{p,q}(\Omega),$$

while by the embedding $W_0^{1,p}(\Omega) \subset W_0^{1,p}(\mathbb{R}^N)$ we get

$$T_{N,p} \leq \lambda_{p,p^*}(\Omega),$$

thus we can conlude that

$$T_{N,p} \leq \liminf_{q \to p^*} \lambda_{p,q}(\Omega).$$

To prove the limsup inequality, let $r > 0$ and $x_0 \in \Omega$ be such that the ball $B_r(x_0)$ is contained in Ω, which is always possible since Ω is open. Without loss of generality, we can suppose that $r = 1$ and $x_0 = 0$. By recalling the definition (6.1) of U, we define

$$u_n(x) = \Big(U(n\,|x|) - U(n)\Big)_+, \qquad n \in \mathbb{N},$$

which belongs to $W^{1,p}_0(\Omega)$. We then observe that

$$\limsup_{q\to p^*} \lambda_{p,q}(\Omega) \le \limsup_{q\to p^*} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_n|^q\,dx\right)^{\frac{p}{q}}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_n|^{p^*}\,dx\right)^{\frac{p}{p^*}}}, \qquad n\in\mathbb{N}.$$

By definition

$$\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx = n^{p-N} \int_{B_n(0)} \left|U'(|y|)\right|^p\,dy,$$

and

$$\left(\int_{\mathbb{R}^N} |u_n|^{p^*}\,dx\right)^{\frac{p}{p^*}} = n^{p-N} \left(\int_{B_n(0)} |U(|y|)-U(n)|^{p^*}\,dx\right)^{\frac{p}{p^*}},$$

Thus we get

$$\limsup_{q\to p^*} \lambda_{p,q}(\Omega) \le \frac{\displaystyle\int_{B_n(0)} |U'(|x|)|^p\,dx}{\left(\displaystyle\int_{B_n(0)} |U(|y|)-U(n)|^{p^*}\,dx\right)^{\frac{p}{p^*}}},$$

then by taking the limit as n goes to ∞ we conclude. □

About the limit constant μ_{p,p^*} we have the following preliminary result.

Lemma 6.2. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, then*

$$\lim_{q\nearrow p^*} \mu_{p,q}(\Omega) = \mu_{p,p^*}(\Omega). \tag{6.3}$$

In particular, if Ω is convex we get

$$\mu_{p,p^*}(\Omega) \le T_{N,p}. \tag{6.4}$$

Proof. The equality (6.6) can be proved along the same lines as Lemma 2.3.

To prove the estimate (6.4) is sufficient to take the limit on both sides of (3.1) and use (6.6) and (6.2). □

The simple estimate (6.4) can indeed be enhanced.

Proposition 6.3. *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded convex set. The following estimate holds true*

$$\mu_{p,p^*}(\Omega) \le \left(\inf_{x_0\in\partial\Omega} \gamma_\Omega(x_0)\right)^{1-\frac{p}{p^*}} T_{N,p},$$

where for every $x_0\in\partial\Omega$ the quantity $\gamma_\Omega(x_0)$ is the blow-up measure of Ω at x_0, i.e.

$$\gamma_\Omega(x_0) = \lim_{r\to+\infty} \frac{|r\,(\Omega-x_0)\cap B_1(0)|}{\omega_N}.$$

Proof. We define

$$u_n(x) = \begin{cases} U(n^2\,|x|) - U(n), & \text{if } |x| < 1/n, \\ 0, & \text{otherwise,} \end{cases}$$

where U is still defined by (6.1). For every $n \in \mathbb{N}$, the function u_n is radially symmetric decreasing, supported in the ball $B_{1/n}(0)$. For $x_0 \in \partial\Omega$ we define

$$\Omega_n = \left\{ x \in \Omega \,:\, |x - x_0| < \frac{1}{n} \right\},$$

and consider the functions

$$\varphi_n(x) = u_n(x - x_0) \cdot 1_{\Omega_n}.$$

For every $n \in \mathbb{N}$ there exists a unique $t_n \in \mathbb{R}$ such that

$$\int_\Omega |\varphi_n - t_n|^{p^*-2}\,(\varphi_n - t_n)\,dx = 0.$$

and we have $t_n = o(1)$ as n goes to ∞. Then we get

$$\mu_{p,p^*}(\Omega) \le \lim_{n\to\infty} \frac{\displaystyle\int_\Omega |\nabla \varphi_n|^p\,dx}{\left(\displaystyle\int_\Omega |\varphi_n - t_n|^{p^*}\,dx\right)^{\frac{p}{p^*}}} = \lim_{n\to\infty} \frac{\displaystyle\int_\Omega |\nabla \varphi_n|^p\,dx}{\left(\displaystyle\int_\Omega |\varphi_n|^{p^*}\,dx\right)^{\frac{p}{p^*}}}, \tag{6.5}$$

where we used that

$$\lim_{n\to\infty} \frac{\displaystyle\int_\Omega |\varphi_n - t_n|^{p^*}\,dx}{\displaystyle\int_\Omega |\varphi_n|^{p^*}\,dx} = 1$$

With a change of variables, we get

$$\int_\Omega |\nabla \varphi_n|^p\,dx = n^{2\,(p-N)} \int_{n^2\,(\Omega - x_0)\cap B_n(0)} |U'(|y|)|^p\,dy,$$

and

$$\int_\Omega |\varphi_n|^{p^*}\,dx = n^{-2\,N} \int_{n^2\,(\Omega - x_0)\cap B_n(0)} |U(|y|) - U(n)|^{p^*}\,dy.$$

By using these in (6.5), we get

$$\mu_{p,p^*}(\Omega) \le \lim_{n\to\infty} \frac{\displaystyle\int_{n^2\,(\Omega - x_0)\cap B_n(0)} |U'(|y|)|^p\,dy}{\left(\displaystyle\int_{n^2\,(\Omega - x_0)\cap B_n(0)} |U(|y|) - U(n)|^{p^*}\,dy\right)^{\frac{p}{p^*}}}. \tag{6.6}$$

We then observe that by convexity of Ω, the sets $n\,(\Omega - x_0) \cap B_1(0)$ have uniformly bounded measures and perimters. Then we get that $n\,(\Omega - x_0) \cap B_1(0)$ converges (up to a subsequence) in the sense of characteristic functions to a set $\mathcal{V}$. Here $\mathcal{V} \subset B_1(0)$ has measure

$$|\mathcal{V}| = \gamma_\Omega(x_0)\,\omega_N,$$

and is a conical set in the following sense

$$\text{if} \quad x \in \mathcal{V} \quad \text{then} \quad t\,x \in \mathcal{V} \quad \text{for every } t \in \left(0, \frac{1}{|x|}\right).$$

This in turn implies that $n^2\,(\Omega - x_0) \cap B_n(0)$ converges to the infinite cone centered at the origin and generated by $\mathcal{V}$, i.e.

$$\left\{x \in \mathbb{R}^N \setminus \{0\}\,:\, \frac{x}{|x|} \in \partial\mathcal{V}\right\}.$$

By using this information in (6.6), the radial symmetry of the functions involved gives the desired result □

Observe that for a convex set we have $\gamma(x_0) \le 1/2$ for every $x_0 \in \partial\Omega$. Moreover, if the convex set is C^1, then $\gamma(x_0) = 1/2$ for every $x_0 \in \Omega$. In this case, we have the following consequence.

Corollary 6.4. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex C^1 set. Then*

$$\mu_{p,p^*}(\Omega) \le \left(\frac{1}{2}\right)^{1-\frac{p}{p^*}}\, T_{N,p}.$$

6.2. **Conformal case.** We consider the conformal case, i.e. we take $p = N$ in (3.1) which gives

$$\mu_{N,q}(\Omega) < \lambda_{N,q}(B)\,\left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)}\right)^{\frac{N^2}{q}}, \tag{6.7}$$

where B is any $N-$dimensional ball. In what follows we set

$$\alpha_N = N(N\omega_N)^{\frac{1}{N-1}}.$$

We have the following preliminary result.

Proposition 6.5. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. There exists a dimensional constant β_N such that we have*

$$\left(\frac{N}{N-1}\,\beta_N\, e\right)^{N-1} \le \lim_{q\to\infty} q\,\mu_{N,q}(\Omega) \le \left(\frac{N}{N-1}\,\alpha_N\, e\right)^{N-1}.$$

For β_N we have the estimate

$$\frac{\alpha_N}{2^{1/(N-1)}} \le \beta_N \le \alpha_N.$$

Proof. To prove the upper bound, it is sufficient to multiply (6.7) by q and then use that

$$\lim_{q\to\infty} q\,\lambda_{N,q}(B) = \left(\frac{N}{N-1}\,\alpha_N\, e\right)^{N-1},$$

which follows from[6] [17, Lemma 2.2]. To prove the lower bound, we can adapt the proof of [17, Lemma 2.1]. First of all, we recall that in [5] it is shown that

$$\sup_{u\in W^{1,N}(\Omega)}\left\{\int_\Omega \exp\left(\frac{\alpha_N}{2^{1/(N-1)}}\,|u-u_\Omega|^{N'}\right)\,dx\ :\ \int_\Omega |\nabla u|^N\,dx\le 1\right\}<+\infty,$$

where u_Ω denotes the mean of u over Ω. This implies that there exists a maximal constant β_N such that the quantity

$$M_{\mathcal{N}}(\Omega)=\sup_{W^{1,N}(\Omega)}\left\{\min_{t\in\mathbb{R}}\int_\Omega \exp\left(\beta_N\,|u-t|^{N'}\right)\,dx\ :\ \int_\Omega |\nabla u|^N\,dx\le 1\right\},$$

stays finite and of course we have

$$\beta_N\ge \frac{\alpha_N}{2^{1/(N-1)}},$$

since $t=u_\Omega$ is always an admissible competitor. For every $q\ge N$, let $u_q\in W^{1,N}(\Omega)$ be a function achieving $\mu_{N,q}(\Omega)$, normalized by

$$\int_\Omega |\nabla u_q|^N\,dx=1,\qquad q\ge N.$$

We also set t_q to the the minimizer of

$$t\mapsto \int_\Omega \exp\left(\alpha_N\,|u_q-t|^{N'}\right)\,dx,$$

then we have

$$\begin{aligned}\min_{t\in\mathbb{R}}\int_\Omega |u_q-t|^q\,dx\le\int_\Omega |u_q-t_q|^q\,dx&=\alpha_N^{-\frac{q}{N'}}\int_\Omega \left(\alpha_N\,|u_q-t_q|^{N'}\right)^{\frac{q}{N'}}\,dx\\&\le\Gamma\left(\frac{q}{N'}+1\right)\int_\Omega \exp\left(\alpha_N\,|u_q-t_q|^{N'}\right)\,dx\\&\le\Gamma\left(\frac{q}{N'}+1\right)\,M_{\mathcal{N}}(\Omega),\end{aligned}$$

which in turn implies

$$\frac{1}{\Gamma\left(\dfrac{q}{N'}+1\right)}\,\frac{1}{M_{\mathcal{N}}(\Omega)}\le\frac{1}{\displaystyle\min_{t\in\mathbb{R}}\int_\Omega |u_q-t|^q\,dx}=\frac{\left(\displaystyle\int_\Omega |\nabla u_q|^N\,dx\right)^{\frac{q}{N}}}{\displaystyle\min_{t\in\mathbb{R}}\int_\Omega |u_q-t|^q\,dx},$$

that is

$$q\left(\frac{1}{\Gamma\left(\dfrac{q}{N'}+1\right)}\,\frac{1}{M_{\mathcal{N}}(\Omega)}\right)^{\frac{N}{q}}\le q\,\mu_{N,q}(\Omega).$$

By using Stirling formula, we can conclude. □

[6] The result in [17] is for $N=2$, but the very same argument can be easily adapted for a geneal $N\ge 2$.

For completeness, we give the following technical result which shows that the maximization problem defining $M_{\mathcal{N}}(\Omega)$ is unchanged if we replace $W^{1,N}(\Omega)$ by any dense subset.

Lemma 6.6. *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded convex set. If $X(\Omega)$ is a dense subset of $W^{1,N}(\Omega)$, then*

$$M_{\mathcal{N}}(\Omega)=\sup_{X(\Omega)}\left\{\min_{t\in\mathbb{R}}\int_\Omega \exp\left(\beta_N\,|u-t|^{N'}\right)\,dx\,:\,\int_\Omega |\nabla u|^N\,dx=1\right\}.$$

Proof. Let us call

$$M'_{\mathcal{N}}(\Omega)=\sup_{X(\Omega)}\left\{\min_{t\in\mathbb{R}}\int_\Omega \exp\left(\beta_N\,|u-t|^{N'}\right)\,dx\,:\,\int_\Omega |\nabla u|^N\,dx=1\right\},$$

then of course we have $M'_{\mathcal{N}}(\Omega)\le M_{\mathcal{N}}(\Omega)$. In order to prove the reverse inequality, for every $\varepsilon>0$ let $u\in W^{1,N}(\Omega)$ be an admissible function such that

$$\mathcal{M}_N(\Omega)-\varepsilon\le\int_\Omega \exp\left(\beta_N\,|u-t_u|^{N'}\right)\,dx,$$

where t_u attains the minimum of

$$\int_\Omega \exp\left(\beta_N\,|u_\varepsilon-t|^{N'}\right)\,dx.$$

Let $\{u_n\}_{n\in\mathbb{N}}\subset X(\Omega)$ be a sequence strongly converging to u, thus in particular

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^N\,dx=\int_\Omega |\nabla u|^N\,dx,$$

and u_n converges almost everywhere in Ω to u (up to a subsequence), then we define the new sequence

$$v_n=\frac{u_n}{\|\nabla u_n\|_{L^N(\Omega)}},\qquad n\in\mathbb{N}.$$

Let $t_n\in\mathbb{R}^N$ be such that

$$\int_\Omega \exp\left(\beta_N\,|v_n-t_n|^{N'}\right)\,dx=\min_{t\in\mathbb{R}}\int_\Omega \exp\left(\beta_N\,|v_n-t|^{N'}\right)\,dx\le M_{\mathcal{N}}(\Omega).$$

Observe that we have

$$\begin{aligned}M_{\mathcal{N}}(\Omega)\ge\int_\Omega \exp\left(\beta_N\,|v_n-t_n|^{N'}\right)&\ge|\Omega|+\beta_N\int_\Omega |v_n-t_n|^{N'}\,dx\\&\ge|\Omega|+\frac{\beta_N}{2^{N'-1}}\,|\Omega|\,|t_n|^{N'}-\beta_N\int_\Omega |v_n|^{N'}\,dx,\end{aligned}$$

which implies that the sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ is bounded, since

$$\lim_{n\to\infty}\int_\Omega |v_n|^{N'}\,dx=\int_\Omega |u|^{N'}\,dx<+\infty.$$

Then there exists $\widetilde{t} \in \mathbb{R}$ such that t_n converges (up to a subsequence) to $\widetilde{t}$. By Fatou Lemma we then get

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega \exp\left(\beta_N\,|v_n - t_n|^{N'}\right)\,dx &\geq \int_\Omega \exp\left(\beta_N\,|u - \widetilde{t}|^{N'}\right)\,dx \\ &\geq \int_\Omega \exp\left(\beta_N\,|u - t_u|^{N'}\right)\,dx \geq M_{\mathcal{N}}(\Omega) - \varepsilon,\end{aligned}$$

thanks to the minimality of t_u. Since the sequence $\{v_n\}_{n\in\mathbb{N}}$ is admissible for the problem defining $M'_{\mathcal{N}}(\Omega)$, we thus get

$$M'_{\mathcal{N}}(\Omega) \geq M_{\mathcal{N}} - \varepsilon.$$

which in turn gives the desired result, thanks to the arbitrariness of $\varepsilon > 0$. □

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